Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

THE MODIFIED REMAINDERS OBTAINED IN FINDING THE HIGHEST COMMON FACTOR OF TWO POLYNOMIALS.

BY A. J. PELL AND R. L. GORDON.

E. B. Van Vleck[*] has shown how to simplify the computation of the remainders obtained in the process of finding the highest common factor of two polynomials: $a_0x^n + a_1x^{n-1} + \cdots + a_n$ and $b_0x^n + b_1x^{n-1} + \cdots + b_n$. The simplification consists in expressing the remainders as minors of a single determinant R (see page 163) which, on account of its construction, lends itself easily to computation. If the *first* coefficients of some of the remainders are zero, the results stated by Van Vleck must be modified. In this paper we determine by direct method the exact expression of the remainders.

In the derivation of this expression we need to transform two determinants some of whose rows are made up of elements which are two-rowed determinants. We give these transformations in the form of two lemmas, which are easy to prove.

LEMMA 1. *If* λ *and* μ *are positive integers, and* $\gamma_i = \begin{vmatrix} \beta_0 & \beta_{i+1} \\ \alpha_0 & \alpha_{i+1} \end{vmatrix}$ *for* $i = 0, 1, \cdots, 2\lambda + \mu + 1$, *then*

$$\left|\begin{array}{ccccccccccc}
\gamma_0 & \gamma_1 & \gamma_2 & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \gamma_{2\lambda+\mu+1} \\
\beta_0 & \beta_1 & \beta_2 & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \beta_{2\lambda+\mu+1} \\
0 & \gamma_0 & \gamma_1 & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \gamma_{2\lambda+\mu} \\
0 & \beta_0 & \beta_1 & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \beta_{2\lambda+\mu} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & \cdots & \gamma_0 & \gamma_1 & \gamma_2 & \cdots & \cdot & \cdot & \gamma_{\lambda+\mu+1} \\
0 & 0 & 0 & \cdots & \beta_0 & \beta_1 & \beta_2 & \cdots & \cdot & \cdot & \beta_{\lambda+\mu+1} \\
0 & 0 & 0 & \cdots & 0 & \beta_0 & \beta_1 & \cdots & \cdot & \cdot & \beta_{\lambda+\mu} \\
0 & 0 & 0 & \cdots & 0 & 0 & \beta_0 & \cdots & \cdot & \cdot & \beta_{\lambda+\mu-1} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \beta_0 & \cdots & \beta_{\lambda+1}
\end{array}\right|$$

---

[*] E. B. Van Vleck, "On the determination of a series of Sturm's functions by the calculation of a single determinant," *Annals of Mathematics*, 1899, pp. 1–13.

$$= (-1)^{\lambda+1}\beta_0^{\lambda}\begin{vmatrix}
\alpha_0 & \alpha_1 & \alpha_2 & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \alpha_{2\lambda+\mu+2} \\
\beta_0 & \beta_1 & \beta_2 & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \beta_{2\lambda+\mu+2} \\
0 & \alpha_0 & \alpha_1 & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \alpha_{2\lambda+\mu+1} \\
0 & \beta_0 & \beta_1 & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \beta_{2\lambda+\mu+1} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & \cdots & \alpha_0 & \alpha_1 & \alpha_2 & \cdots & \cdot & \cdot & \alpha_{\lambda+\mu+2} \\
0 & 0 & 0 & \cdots & \beta_0 & \beta_1 & \beta_2 & \cdots & \cdot & \cdot & \beta_{\lambda+\mu+2} \\
0 & 0 & 0 & \cdots & 0 & \beta_0 & \beta_1 & \cdots & \cdot & \cdot & \beta_{\lambda+\mu+1} \\
0 & 0 & 0 & \cdots & 0 & 0 & \beta_0 & \cdots & \cdot & \cdot & \beta_{\lambda+\mu} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \beta_0 & \cdots & \beta_{\lambda+1}
\end{vmatrix}.$$

LEMMA 2. *If* λ *and* μ *are positive integers,* $\begin{vmatrix}\beta_0\beta_i \\ \alpha_0\alpha_i\end{vmatrix} = 0$ *for* $i = 1, 2, \cdots, \mu - 1$, *and* $\gamma_{i-\mu} = \begin{vmatrix}\beta_0\beta_i \\ \alpha_0\alpha_i\end{vmatrix}$ *for* $i = \mu, \mu + 1, \cdots, 2\lambda + 2\mu + 1$, *then*

$$\begin{vmatrix}
\gamma_0 & \gamma_1 & \gamma_2 & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \gamma_{2\lambda+\mu+1} \\
\beta_0 & \beta_1 & \beta_2 & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \beta_{2\lambda+\mu+1} \\
0 & \gamma_0 & \gamma_1 & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \gamma_{2\lambda+\mu} \\
0 & \beta_0 & \beta_1 & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \beta_{2\lambda+\mu} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & \cdots & \gamma_0 & \gamma_1 & \gamma_2 & \cdots & \cdot & \cdot & \gamma_{\lambda+\mu+1} \\
0 & 0 & 0 & \cdots & \beta_0 & \beta_1 & \beta_2 & \cdots & \cdot & \cdot & \beta_{\lambda+\mu+1} \\
0 & 0 & 0 & \cdots & 0 & \gamma_0 & \gamma_1 & \cdots & \cdot & \cdot & \gamma_{\lambda+\mu} \\
0 & 0 & 0 & \cdots & 0 & 0 & \gamma_0 & \cdots & \cdot & \cdot & \gamma_{\lambda+\mu-1} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \gamma_0 & \cdots & \gamma_{\lambda+1}
\end{vmatrix}$$

$$= (-1)^{\mu(\lambda+1)}(-1)^{1+2+\cdots+\mu}\beta_0^{\lambda+1}\begin{vmatrix}
\alpha_0 & \alpha_1 & \alpha_2 & \cdots & \cdot & \cdot & \alpha_{2\lambda+2\mu+1} \\
\beta_0 & \beta_1 & \beta_2 & \cdots & \cdot & \cdot & \beta_{2\lambda+2\mu+1} \\
0 & \alpha_0 & \alpha_1 & \cdots & \cdot & \cdot & \alpha_{2\lambda+2\mu} \\
0 & \beta_0 & \beta_1 & \cdots & \cdot & \cdot & \beta_{2\lambda+2\mu} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & \cdots & \alpha_0 & \cdots & \alpha_{\lambda+\mu+1} \\
0 & 0 & 0 & \cdots & \beta_0 & \cdots & \beta_{\lambda+\mu+1}
\end{vmatrix}.$$

The following theorem gives the exact expression of the remainders in terms of the minors of R, where

$$R = \begin{vmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} & a_n & 0 & \cdots & 0 \\ b_0 & b_1 & b_2 & \cdots & b_{n-1} & b_n & 0 & \cdots & 0 \\ 0 & a_0 & a_1 & \cdots & a_{n-2} & a_{n-1} & a_n & \cdots & 0 \\ 0 & b_0 & b_1 & \cdots & b_{n-2} & b_{n-1} & b_n & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & a_0 & a_1 & a_2 & \cdots & a_n \\ 0 & 0 & 0 & \cdots & b_0 & b_1 & b_2 & \cdots & b_n \end{vmatrix}$$

Theorem. Let

(*A*) $$a_0x^n + a_1x^{n-1} + \cdots + a_n,$$

and

(*B*) $$b_0x^n + b_1x^{n-1} + \cdots + b_n$$

be two polynomials of the nth degree. Modify the process of finding the highest common factor of (*A*) *and* (*B*) *by taking at each stage the negative of the remainder. Let the ith modified remainder be*

$R^{(i)}$ $$r_0{}^{(i)}x^{m_i} + r_1{}^{(i)}x^{m_i-1} + \cdots + r_{m_i}{}^{(i)},$$

where $(m_i + 1)$ *is the degree of the preceding remainder, and where the first* $(p_i - 1)$ *coefficients of* $R^{(i)}$ *are zero, and the* p_i*th coefficient* $\rho_i = r_{p_i-1}{}^{(i)}$ *is different from zero. Then for* $k = 0, 1, \cdots, m_i$, *the coefficients* $r_k{}^{(i)}$ *are given by**

$$(1)\quad r_k{}^{(i)} = \frac{(-1)^{u_{i-1}}(-1)^{u_{i-2}}\cdots(-1)^{u_1}(-1)^{v_{i-1}}}{\rho_{i-1}^{p_{i-1}+1}\rho_{i-2}^{p_{i-2}+p_{i-1}}\cdots\rho_1^{p_1+p_2}\rho_0^{p_1}}$$

$$\times\begin{vmatrix} a_0 & a_1 & a_2 & \cdots & \cdot & \cdot & \cdot & a_{2v_{i-1}} & a_{2v_{i-1}+1+k} \\ b_0 & b_1 & b_2 & \cdots & \cdot & \cdot & \cdot & b_{2v_{i-1}} & b_{2v_{i-1}+1+k} \\ 0 & a_0 & a_1 & \cdots & \cdot & \cdot & \cdot & a_{2v_{i-1}-1} & a_{2v_{i-1}+k} \\ 0 & b_0 & b_1 & \cdots & \cdot & \cdot & \cdot & b_{2v_{i-1}-1} & b_{2v_{i-1}+k} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & a_0 & a_1 & \cdots & a_{v_{i-1}} & a_{v_{i-1}+1+k} \\ 0 & 0 & 0 & \cdots & b_0 & b_1 & \cdots & b_{v_{i-1}} & b_{v_{i-1}+1+k} \end{vmatrix},$$

where $u_{i-1} = 1 + 2 + \cdots + p_{i-1}$, *and* $v_{i-1} = p_1 + p_2 + \cdots + p_{i-1}$.

Divide (*A*) by (*B*) and the remainder with sign changed may be expressed as follows:

---

* It is to be understood in (1) that $\rho_0 = b_0$, $p_0 = 0$, and that $a_i = b_i = 0$ for $i > n$.

$$R^{(1)} = \frac{1}{b_0}\left\{ \begin{vmatrix} a_0 & a_1 \\ b_0 & b_1 \end{vmatrix} x^{n-1} + \begin{vmatrix} a_0 & a_2 \\ b_0 & b_2 \end{vmatrix} x^{n-2} + \cdots + \begin{vmatrix} a_0 & a_n \\ b_0 & b_n \end{vmatrix} \right\}.$$

Hence the coefficients of $R^{(1)}$ satisfy (1) for $i = 1$.

Suppose that (1) is true for $i = l$, then it follows that

$$(2)\quad r_k^{(l+1)} = \frac{(-1)^{u_l}(-1)^{u_{l-1}} \cdots (-1)^{u_2}(-1)^{v_l-p_1}}{\rho_l^{p_l+1}\rho_{l-1}^{p_{l-1}+p_l} \cdots \rho_2^{p_2+p_3}\rho_1^{p_2}}$$

$$\times \begin{vmatrix} \bar{s}_0 & \bar{s}_1 & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot & \bar{s}_{2v_l-2p_1} & \bar{s}_{2v_l-2p_1+1+k} \\ r_{p_1-1}^{(1)} & r_{p_1}^{(1)} & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot & r_{2v_l-p_1-1}^{(1)} & r_{2v_l-p_1+k}^{(1)} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 0 & \bar{s}_0 & \bar{s}_1 & \cdots & & \bar{s}_{v_l-p_1} & \bar{s}_{v_l-p_1+1+k} \\ 0 & 0 & \cdots & 0 & r_{p_1-1}^{(1)} & r_{p_1}^{(1)} & \cdots & & r_{v_l-1}^{(1)} & r_{v_l+k}^{(1)} \end{vmatrix},$$

where the $\bar{s}_i$ are the coefficients of the remainder of the next to the last stage in dividing B by $R^{(1)}$. The $\bar{s}_i$ may be expressed as two-rowed determinants:

$$\bar{s}_i = \frac{1}{\rho_1}\begin{vmatrix} r_{p_1-1}^{(1)} & r_{i+p_1}^{(1)} \\ \bar{\bar{s}}_0 & \bar{\bar{s}}_{i+1} \end{vmatrix},$$

and hence Lemma 1 may be applied to the determinant in (2), where $\lambda = v_l - p_1$ and $\mu = 0$. A determinant of the same character as in (2) is obtained, for the $\bar{\bar{s}}_i$ are also expressible as $1/\rho_1$ times two-rowed determinants:

$$\bar{\bar{s}}_i = \frac{1}{\rho_1}\begin{vmatrix} r_{p_1-1}^{(1)} & r_{i+p_1}^{(1)} \\ \bar{\bar{s}}_0 & \bar{\bar{s}}_{i+1} \end{vmatrix}.$$

Lemma 1 may be applied again, with $\lambda = v_l - p_1$ and $\mu = 1$. The process may be repeated, and after applying Lemma 1 p_1 times,* a determinant in terms of b_i and $r_i^{(1)}$ is obtained. The factor outside the determinant becomes multiplied by

$$\left(\frac{(-1)^{v_l-p_1+1}\rho_1^{v_l-p_1}}{\rho_1^{v_l-p_1+1}}\right)^{p_1},$$

and the determinant for $r_k^{(l+1)}$ takes the form

---

* Some of the partial quotients may be zero, that is, the first elements in some sets of s's may be zero. In that case the elements of one set of s's are equal to the elements of the next set, with the subscripts advanced by one.

$$
\begin{vmatrix}
b_0 & b_1 & b_2 & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot & b_{2v_l-2p_1+1} & b_{2v_l-2p_1+2+k} \\
r^{(1)}_{p_1-1} & r^{(1)}_{p_1} & r^{(1)}_{p_1+1} & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot & r^{(1)}_{2v_l-p_1} & r^{(1)}_{2v_l-p_1+1+k} \\
0 & b_0 & b_1 & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot & b_{2v_l-2p_1} & b_{2v_l-2p_1+1+k} \\
0 & r^{(1)}_{p_1-1} & r^{(1)}_{p_1} & \cdots & \cdot & \cdot & \cdot & \cdot & \cdot & r^{(1)}_{2v_l-p_1-1} & r^{(1)}_{2v_l-p_1+k} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & \cdots & b_0 & \cdot & \cdot & \cdot & \cdots & b_{v_l-p_1+1} & b_{v_l-p_1+2+k} \\
0 & 0 & 0 & \cdots & r^{(1)}_{p_1-1} & \cdot & \cdot & \cdot & \cdots & r^{(1)}_{v_l} & r^{(1)}_{v_l+1+k} \\
0 & 0 & 0 & \cdots & 0 & r^{(1)}_{p_1-1} & \cdot & \cdot & \cdots & r^{(1)}_{v_l-1} & r^{(1)}_{v_l+k} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & \cdots & 0 & 0 & \cdots & r^{(1)}_{p_1-1} & \cdots & r^{(1)}_{v_l-p_1} & r^{(1)}_{v_l-p_1+1+k}
\end{vmatrix}. \tag{3}
$$

Now, $r_i^{(1)} = 0$ for $i = 0, 1, \cdots, (p_i - 2)$, and

$$
r_i^{(1)} = -\frac{1}{b_0}\begin{vmatrix} b_0 & b_{i+1} \\ a_0 & a_{i+1} \end{vmatrix}
$$

for $i = p_1 - 1, p_1, \cdots$, so that if each pair of two consecutive b_i and $r_i^{(1)}$ rows in (3) are interchanged, a determinant of the form in Lemma 2 is obtained. Hence Lemma 2 may be applied with $\lambda = v_l - p_1$ and $\mu = p_1$. The result is that the factor outside the determinant is multiplied by

$$
\frac{(-1)^{v_l-p_1+1}(-1)^{p_1(v_l-p_1+1)}(-1)^{u_1}\rho_0^{v_l-p_1+1}}{(-\rho_0)^{v_l+1}}.
$$

and the determinant becomes one with $p_l + p_{l-1} + \cdots + p_1 + 1$ pairs of a_i and b_i rows. After simplifying the outside factor the following is obtained:

$$
r_k^{(l+1)} = \frac{(-1)^{u_l}(-1)^{u_{l-1}}\cdots(-1)^{u_1}(-1)^{v_l}}{\rho_l^{p_l+1}\rho_{l-1}^{p_{l-1}+p_l}\cdots\rho_1^{p_1+p_2}\rho_0^{p_1}}
$$

$$
\times\begin{vmatrix}
a_0 & a_1 & a_2 & \cdots & \cdot & \cdot & \cdot & a_{2v_l} & a_{2v_l+1+k} \\
b_0 & b_1 & b_2 & \cdots & \cdot & \cdot & \cdot & b_{2v_l} & b_{2v_l+1+k} \\
0 & a_0 & a_1 & \cdots & \cdot & \cdot & \cdot & a_{2v_l-1} & a_{2v_l+k} \\
0 & b_0 & b_1 & \cdots & \cdot & \cdot & \cdot & b_{2v_l-1} & b_{2v_l+k} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & \cdots & a_0 & a_1 & \cdots & a_{v_l} & a_{v_l+1+k} \\
0 & 0 & 0 & \cdots & b_0 & b_1 & \cdots & b_{v_l} & b_{v_l+1+k}
\end{vmatrix}.
$$

This is the form of (1) for $i = l + 1$, and the theorem is proved.

In case $b_0 > 0$, and $p_i = 1$ for $i = 1, 2, \cdots, n$, that is, when the first coefficients of all the remainders are different from zero, the factor outside the determinant is positive, and the coefficients of the ith remainder are given except for the positive factor by the minors of R formed from the first $2i$ rows by associating the elements in the first $2i - 1$ columns with each of the other columns. This is the rule which Van Vleck* gives as holding for any case, but if, for example, $p_1 = 2$, the coefficients of the second modified remainder $R^{(2)}$ are given by the 6-rowed minors formed from the first 6 rows of R, and the sign factor is

$$\frac{(-1)^{1+2}(-1)^2}{b_0^2 \rho_1^3},$$

which is negative if ρ_1 is positive.

If (A) is of degree n, and (B) of degree $n - p_0$, the coefficients of the ith modified remainder

$$R^{(i)} \qquad r_0^{(i)} x^{m_i} + r_1^{(i)} x^{m_i - 1} + \cdots + r_{m_i}^{(i)}$$

are given for $k = 0, 1, 2, \cdots, m_i$, by

$$r_k^{(i)} = \frac{(-1)^{v_{i-1}}(-1)^{u_{i-1}}(-1)^{u_{i-2}} \cdots (-1)^{u_0}(-1)^{(p_0/2)(2v_{i-1}-p_0+1)}}{\rho_{i-1}^{p_{i-1}+1}\,\rho_{i-2}^{p_{i-2}+p_{i-1}} \cdots \rho_1^{p_1+p_2}\,\rho_0^{p_0+p_1}}$$

$$\times \begin{vmatrix} a_0 & a_1 & a_2 \cdots & \cdot & \cdot & \cdot & \cdot & a_{2v_{i-1}-p_0} & a_{2v_{i-1}-p_0+1+k} \\ b_0 & b_1 & b_2 \cdots & \cdot & \cdot & \cdot & \cdot & b_{2v_{i-1}-p_0} & b_{2v_{i-1}-p_0+1+k} \\ 0 & a_0 & a_1 \cdots & \cdot & \cdot & \cdot & \cdot & a_{2v_{i-1}-p_0-1} & a_{2v_{i-1}-p_0+k} \\ 0 & b_0 & b_1 \cdots & \cdot & \cdot & \cdot & \cdot & b_{2v_{i-1}-p_0-1} & b_{2v_{i-1}-p_0+k} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots a_0 & a_1 & \cdots & & a_{v_{i-1}} & a_{v_{i-1}+1+k} \\ 0 & 0 & 0 & \cdots b_0 & b_1 & \cdots & & b_{v_{i-1}} & b_{v_{i-1}+1+k} \\ 0 & 0 & 0 & \cdots 0 & b_0 & \cdots & & b_{v_{i-1}-1} & b_{v_{i-1}+k} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots 0 & \cdots & b_0 & \cdots & b_{v_{i-1}-p_0} & b_{v_{i-1}-p_0+1+k} \end{vmatrix},$$

where $u_i = 1 + 2 + \cdots + p_i$ and $v_i = p_0 + p_1 + \cdots + p_i$.

---

* L. c., pp. 3, 4.

Mt. Holyoke College.